

Review

Optimization of input in animal production: A linear programming approach to the ration formulation problem

¹M. Nabasirye*, ²J.Y.T. Mugisha, ¹F. Tibayungwa and ¹C. C. Kyarisiima

¹Department of Agricultural Production, Makerere University, P. O. Box 7062 Kampala, Uganda

²Department of Mathematics, Makerere University, P. O. Box 7062 Kampala, Uganda

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With cost of feed being a major factor in the overall cost of production, it is imperative that those engaged in the training of nutritionists and animal producers give trainees adequate knowledge and skills to make informed decisions. This paper shows how to formulate a least-cost diet in linear programming. The computer output is discussed and the importance of proper interpretation of the sensitivity report is particularly emphasized based on Microsoft Excel[®] Solver output format.

Keywords: Linear programming, least cost ration, sensitivity analysis, optimal solution.

INTRODUCTION

Feed costs account for 50-80% of the total costs in animal production (Pond et al., 1995). In pig production costs account for 75-85% (Gallenti, 1997), over 60% in poultry production (Rose, 1997), and in milk production feed costs are the largest expense (Bath, 1985). Therefore procedures that reduce feed costs are likely to increase net incomes in animal agriculture. Ration formulation is one of the areas that one can use to reduce on the cost of feed.

Several methods have been used in formulating and balancing rations. These include the Pearson Square, simultaneous equations, trial and error, and linear programming (LP). If we choose to formulate and mix feed aiming at a nutritionally balanced and adequate diet while keeping the cost at the minimum then LP is the only candidate. Even when LP is used, what our farmers and feed mill operators get from nutritionists are simply the proportions in which to mix the ingredients. The "what-if" scenarios are never part of the package.

The trend nowadays in the job market is such that employers require employees who are numerate and computer literate. As observed by Oakshott (1997), the

ability to use computer software, and to be able to make sensible recommendations based on the output from the software, is an essential prerequisite to the success of employee and ultimately the employer. Because many of these programs involve the use or the development of models, an understanding of how these models work, as well as their strengths and weaknesses, is an important skill.

Linear programming

If the feasible region is a subset of the non-negative portion of R^n , defined by linear equations and inequalities, and the objective function to be minimized or maximized is linear, then we have a linear programming problem (Meyer, 1985). Selecting the best alternative out of a large number of possibilities is called *optimization*.

Linear programming has been widely used in livestock rations (Lara, 1993); and to formulate diets, nutritionists should be knowledgeable in diet specifications as well as interpretation of results (Pond et al., 1995). One of the outputs of an LP formulation is a sensitivity report. Sensitivity analysis (also known as parametric programming or post-optimality analysis) should be stressed as one of the most important issues in LP and should arise with every model (Render and Stair, 1994).

*Corresponding author email: mnabasirye@agric.mak.ac.ug.
256-414-533580/256-772-519966 . Fax: 256-414-531641.

The rest of this paper is on how to formulate a least-cost diet, and how to utilize the computer output in making informed decisions.

The least-cost ration

Mathematical formulation of the linear problem

Given restrictions (constraints) on the use of a given set of inputs, and the target values to be met, the following assumptions are necessary for a least-cost ration:

- 1- all inputs into the ration are infinitely divisible,
- 2- all the coefficients (objective and activity) are known with certainty,
- 3- the total of all activities equals the sum of individual activities. In algebraic terms, our formulation takes the following form:

$$\text{Minimize } Z = \sum_{j=1}^n C_j x_j$$

$$\text{Subject to } \sum_{j=1}^n a_{ij} x_j \leq (\geq, =) b_i$$

Where Z is total cost of ration

$$\sum_{j=1}^n x_j = q$$

$$x_j \geq 0$$

C_j = cost of ingredient j

x_j = quantity of ingredient j

a_{ij} = quantity of nutrient i in ingredient j

b_i = required amount of nutrient i in the ration

q = weight value.

The constraint type (equality or inequality) depends on the nutrient or the nutrient balance required in the ration. We may need to set a minimum level for a given nutrient, rather than zero, because of its desirable characteristics. Likewise an upper limit may be necessary due to undesirable characteristics or simply to avoid imbalance of nutrients especially if this nutrient costs less than other nutrients. If we denote the lower and upper limits of nutrient i in a unit of the feed as l'_i and u'_i respectively, then we have

$$l'_i q \leq \sum_{j=1}^n a_{ij} x_j \leq u'_i q$$

The constraints can then be modified as follows:

$$\begin{aligned} ql'_1 &\leq a_{11}x_1 + a_{12}x_2 \dots + a_{1n}x_n \leq qu'_1 \\ ql'_2 &\leq a_{21}x_1 + a_{22}x_2 \dots + a_{2n}x_n \leq qu'_2 \\ &\vdots \\ ql'_m &\leq a_{m1}x_1 + a_{m2}x_2 \dots + a_{mn}x_n \leq qu'_m \\ 0 &\leq x_1 \leq u_1 \\ 0 &\leq x_2 \leq u_2 \\ &\vdots \\ 0 &\leq x_n \leq u_n \end{aligned}$$

Assuming a symmetrical probability distribution, formulating rations basing on average nutrient values means meeting nutrient requirement 50% of the time. Likewise rations formulated basing on average nutrient requirements means meeting requirement of 50% of the animals. Ingredient nutrient variation and variation in nutrient requirements need to be accounted for when formulating rations.

By taking nutrient requirements and ingredient nutrients as random variables, we can use the Mathematical Expectation procedures to account for this variability.

The expected (mean) quantity of nutrient in the ration is given by

$$E(R_i) = \sum E(A_{ij})X_j = \sum \mu_{ij}x_j$$

And the variance by

$$V(R_i) = \sum_{j=1}^n v_{ij}x_j^2 + \sum_{j=1}^n \sum_{j'=1}^n cv_{ij,j'}x_jx_{j'} \quad j \neq j'$$

where R_i is the quantity of nutrient i per day

$\mu_{ij} = A_{ij}$ is the expected value of nutrient i in ingredient j

x_j is the j^{th} ingredient

v_{ij} is the variance of A_{ij}

$V()$ denotes the expectation operator for the weighted average.

Since this is a situation of independent random variables, the second term of the variance equation drops out, as the covariances are zero (Freund, 1994), giving the variance as

$$V(R_i) = \sum_{j=1}^n v_{ij}x_j^2 \quad V(R_i) = \sum_{j=1}^n v_{ij}x_j^2$$

If our interest is to have R_i exceed the nutrient requirement (b_i) a prescribed percentage of time (α_i),

$prob(R_i \geq b_i) \geq \alpha_i$, where $1 > \alpha_i > 0$, then we need to transform R_i into a standard normal variable.

From the statistical theory, if R denotes a normally distributed random variable, a new variable, $\phi = (R - E(R)) / \sqrt{V(R)}$, is distributed as a standard normal variable with $E(\phi) = 0$ and $V(\phi) = 1$.

This means

$$prob \left[\left(\frac{R_i - E(R_i)}{\sqrt{V(R_i)}} \right) \geq \left(\frac{b_i - E(R_i)}{\sqrt{V(R_i)}} \right) \right] \geq \alpha_i$$

The first term can be substituted by the standard normal variable to give

$$prob \left[\phi \geq \left(\frac{b_i - E(R_i)}{\sqrt{V(R_i)}} \right) \right] \geq \alpha_i$$

This can now be written in deterministic terms as

$$\sum \mu_{ij} x_j \geq b_i + \phi \sqrt{\sum v_{ij} x_j^2}$$

where second term of the RHS (which is a function of the variance of nutrients in the ration) is taken as the safety factor, and ϕ is the standard normal deviate whose value is determined by the requested probability of success (α_i).

By accounting for nutrient variability we have arrived at a non-linear formulation of the mean and variance. Non-linear programming algorithms are available and very capable of handling this problem. However, if we have to use linear programming then these non-linear components must be linearised. A linear approximation by Rahman and Bender (1971) replaces

$$\sqrt{\sum v_{ij} x_j^2} \text{ with } \sqrt{\sum \delta_{ij} x_j}$$

But this simplification results in a mathematical error (Tozer, 2000; D'Alfonso et al., 1992) of assuming $\sqrt{a} + \sqrt{b}$ is equal to $\sqrt{a+b}$, where a and b are positive real numbers. Nevertheless, as demonstrated by Rahman and Bender (1971) the approximation is acceptable as the inherent bias in this approximation is such that in no case would the actual probability of meeting the requirements be less than the specified value. The probability of success should be determined by carefully considering the benefit foregone if the requirement is not met vis-a-vis the loss if the requirement is exceeded.

Implementation

LINDO (Linear Interactive aNd Discrete Optimizer),

GAMS (General Algebraic Modelling System) and Microsoft Excel® Solver are some of the numerous software packages dedicated to solving linear programming problems. However, Microsoft Excel® is perhaps the most popular spreadsheet used both in business and in universities and as such is very accessible. Second to this, the spreadsheet offers very convenient data entry and editing features which allows the user to gain a greater understanding of how to construct linear programs. For the purposes of this paper, only the output format from Microsoft Excel® Solver is presented.

Interpretation of the computer output

The least-cost formulation is given at a cost indicated in column "Final Value" under "Target Cell" section (Figure1).

The amounts in which to mix the ingredients are given in column "Final Value" under "Adjustable Cells" section. Slack value indicates the magnitude by which the calculated amount deviates from the required amount. Constraints with slack values of zero are said to be active or binding whereas for non-zero are inactive or non-binding. Now that the optimal solution to our LP problem has been found, what next? Recall that this optimal solution was arrived at under deterministic assumptions: prices for ingredients were assumed to be fixed, nutrient composition fixed, nutrient requirements fixed. However, in a real world situation these factors are dynamic. Therefore when we get an optimal solution, we need to see how sensitive our optimum solution is to model assumptions and data changes without having to re-solve the entire problem. This is what sensitivity analysis is about - giving insight into "what-if questions".

Sensitivity analysis

The discussion will focus only on 1) the variation of a unit in the Right Hand Side of constraints and 2) relative variation of optimized target function.

Change in the RHS

From the mathematics of the Simplex method we know (for an optimal basic feasible solution that is degenerate) that

$$z = c_B B^{-1} b - \sum_{j \in R} (z_j - c_j) x_j = w^* b - \sum_{j \in R} (z_j - c_j) x_j$$

where B is an optimal basis for the primal problem and c_B is its associated cost vector,

R is the index set for the non basic variables x_j that may include both slack and structural variables,

Microsoft Excel 11.0 Answer Report					
Worksheet: [Book1]Sheet1					
Report Created: 8/5/2006 21:16:01					
Target Cell (Min)					
Cell	Name	Original Value	Final Value		
Adjustable Cells					
Cell	Name	Original Value	Final Value		
Constraints					
Cell	Name	Cell Value	Formula	Status	Slack

Figure 1. Microsoft Excel 11.0 Answer Report Format

$$x_j \geq 0$$

$$z_j = c_B \mathbf{B}^{-1} \mathbf{a}_j \text{ for each non basic variable.}$$

Since the current basic feasible solution remains feasible with slight perturbation of the z RHS b_i , optimality is maintained. Given z^* as the optimal Objective Value (OV) and \mathbf{B}_i^{-1} as the b column of \mathbf{B}^{-1} , the above equation can be written as a partial differential coefficient of z with respect to b , thus

$$\frac{\partial z^*}{\partial b_i} = c_B \mathbf{B}_i^{-1} = w_i^*$$

(1.13)

This means w_i^* is the rate of change of the objective value per unit increase in the i^{th} RHS value, given that the current non basic values are held at zero (regardless of feasibility). Since $w_i^* \geq 0$, z^* will change (or stay constant) as b_i changes.

Sensitivity analysis report

(Figure 2). The values in the "Shadow Price" column indicate rate of change in the OV as (RHS) changes. It is important to note that shadow prices are dual prices in LINDO. A positive shadow price means that the OV

and the RHS move in the same direction and a negative shadow price means that the OV and RHS move in opposite directions. There may be a need to reduce the OV for economic reasons. Consider a case where the production costs are such that this OV is above the break-even point. Loosening on one of the active requirements would be helpful. The limits within which we can change the RHS are given by values under "Allowable Decrease" and "Allowable Increase". As our interest, in this case, is to cut down the costs, we are only interested in loosening the RHS. The values in column "Shadow Price" are the rate of change in the OV as the RHS changes. However, if the priority is to make a given constraint (requirement) in the ration less marginal, we could raise it by any value up to the "Allowable Increase" value. For every unit increase in constraint (requirement), the OV would be increasing by a corresponding shadow price value. Going beyond the allowable range, the rate of change in the current solution may not hold. Even when one is still within the allowable range, there may be a need to stop increase or decrease of RHS. The level at which to stop would be determined by economics of feed production and the level that is not detrimental to the animal.

Change in the cost vector

Here we are concerned with the cost of one of the variables changing, say, from C_k to C'_k in a given optimal

Microsoft Excel 11.0 Sensitivity Report						
Worksheet: [Book1]Sheet1						
Report Created: 8/5/2006 21:16:02						
Adjustable Cells						
Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
Constraints						
Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease

Figure 2. Microsoft Excel 11.0 Sensitivity Report Format

basic feasible solution. This change in the cost vector can be on:

- ingredients not included in the optimal solution (the non basic variables),
- ingredients included in the optimal solution (basic variables).

Ingredients not included in the optimal solution (the non basic variables)

In this case $Z_k - C_k$ ($Z_k - C_k \leq 0$ is the condition for optimality in a minimization problem) is replaced by $Z_k - C'_k$. But Z_j is not affected for any j since c_B is not changed. However, if the cost of a non basic variable reduces sufficiently enough such that $Z_k - C'_k$ becomes positive, then this ingredient can now be included. The amount by which C_k must reduce for the non basic variable to be used in the model, is given by the $Z_k - C_k$ values indicated in the final simplex tableau.

Alternatively, this value can be read off directly under "Reduced Cost" column. When the cost reduces by the exact value indicated "reduced cost", it does not mean that the inclusion of this ingredient will reduce the OV. Inclusion of this ingredient at this cost means that we now have two optima, the original (without this ingredient) and the new (with this ingredient). Choice between these two optima depends on knowledge of animal nutrition rather than economic reasons. Ingredients included in the optimal solution (basic variables). Consider a situation where the cost of a given ingredient has gone up (other factors constant). This means the OV goes up as well. As the cost of an ingredient goes up, we might use less

and less of it, until a certain level when none of it is used in the optimal solution. But when do we know when the price has become prohibitive. The price of an ingredient can vary over a given range without influencing the optimal ration. Looking at "Adjustable Cells" section we see values for allowable increase and decrease. This is the range within which the cost can vary without changing the optimal solution. By optimal solution staying the same refers to the optimal values of all the variables, including the slack values, but the total cost will increase by a value equivalent to the product "Final Value" of a given ingredient and the difference between original cost and new cost of the ingredient.

Let X_1 be original objective coefficient

X_2 the new objective coefficient

X_3 the final value

C the original OV

then the new OV (NOV) is given by

$$NOV = C + (X_2 - X_1) * X_3$$

It is therefore advisable to recalculate the profit to ascertain that the given increase in cost is cost- effective. However, if the cost of a given ingredient increases by more than the allowable amount, the optimal value of this ingredient should decrease.

As Wendell (1985) notes, the foregoing sensitivity analysis makes it difficult for a decision maker to handle more than one coefficient or term at a time. For a discussion on how to consider simultaneous and independent changes in the objective function and in the RHS, refer to Wendell (1985). This simultaneous and independent approach is desirable in ration formulation analysis, as in nature these simultaneous changes are bound to happen.

CONCLUSION

Applied mathematics is a powerful tool by which we can better conceptualize the nutritional-economic models. However, it should be used within the biological context. It is therefore essential to have solid grounding in animal nutrition in order to make sensible decisions. Central to proper interpretation of an LP output is an understanding of strengths and limitations of these optimization models. Although Microsoft Excel® Solver uses an efficient optimization algorithm, one has to setup the spreadsheet model in an appropriate form adhering to technical restrictions imposed by Solver. It is therefore essential to have adequate knowledge of how these models work for good modelling practice. Fylstra *et al.* (1998) describes some of the common pitfalls in Microsoft Excel® Solver. To take advantage of the benefits of stochastic programming, there is need to have variances and/or standard deviations of nutrients documented alongside the nutrient means in the ingredient data bases during compilation by researchers and nutritionists.

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