Short Communication

Lagrangian Equation for Fluid Mechanics

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Abstract

This paper is intended to build a connection between dynamics and fluid mechanics. Since Lagrangian equations are very useful tools in dynamics now it is discovered that it can be also used in fluid mechanics, so connection can be built through this equation. Here, the lagrangian equation is derived from the momentum equation in fluid mechanics, and then the equation is applied to three different coordinates, Cartesian, cylindrical and spherical. Certainly the application of the equation is not limited to these three different coordinates. This is just for illustration. Many applications are expected.

Keywords: Lagrangian equation; Dynamics; Fluid mechanics; Tools

INTRODUCTION

Fluid mechanics is my major in my graduate study. In my memory when I started to study the fluid mechanics, I learned a complete set of different equations in fluid mechanics. It seems not much related with dynamics. Here I hope that I can get some connections between dynamics and fluid mechanics. In dynamics, Lagrangian equations are very useful tools; many equations are developed with the use of Lagrangian equation. Now based on the momentum equation in fluid mechanics, I derived the Lagrangian equation, so I prove that Lagrangian equation can be also used in fluid mechanics. To simplify the derivation, I started the derivation for incompressible fluid, so a more general form of Lagrangian equation can be further developed. To illustrate the use of the equation, some examples are given. Those are especially in very familiar area, so people can immediately recognize that the equation is working. Momentum equations for inviscid incompressible fluid in Cartesian, cylindrical and spherical coordinates are chosen for the illustration (Figures 1 and 2) [1-4].

DESCRIPTION

Derivation of Lagrangian equation

In dynamics, especially in the Chapter of Lagrange's equations, the position vector r is considered as a function of generalized coordinates q_i and time t.

$$r = r(q_i, t)$$

And velocity $\dot{r}$ is a function of generalized coordinates q_i, generalized velocity $\dot{q}_i$ and time t.

$$\dot{r} = \dot{r}(q_i, \dot{q}_i, t)$$

These notations will be used here in this paper.

In fluid mechanics, for incompressible fluid, when a portion of fluid is observed, following this portion of fluid, the momentum equation can be written as:

$$\rho \frac{DV}{Dt} - F = 0 \qquad (1)$$

Where ρ is density, $\frac{D}{Dt}$ is the substantial derivative, V is velocity and F is force per volume.

Applying the virtual work concept, the equation becomes:

$$\left(\rho \frac{DV}{Dt} - F \right) \cdot dr = 0 \qquad (2)$$

Since the virtual work is done at one instant of time, r is function of generalized coordinates only, so

$$\left(\rho \frac{\mathbf{DV}}{\mathbf{Dt}} - \mathbf{F}\right) \bullet \sum_i \frac{\partial \mathbf{r}}{\partial q_i} \, dq_i = 0 \quad (3)$$

Now equation (3) is to be considered by two parts, first part is

$$\rho \, \frac{\mathbf{DV}}{\mathbf{Dt}} \bullet \frac{\partial \boldsymbol{r}}{\partial q_i} \, dq_i \; = \; \rho \left[\left(\frac{\mathbf{D}}{\mathbf{Dt}} \, \dot{r}\right) \bullet \frac{\partial \mathbf{r}}{\partial q_i}\right] dq_i$$

$$= \; \rho \, \left[\frac{\mathbf{D}}{\mathbf{Dt}}\left(\dot{r} \bullet \frac{\partial \mathbf{r}}{\partial q_i}\right) - \dot{r} \bullet \frac{\mathbf{D}}{\mathbf{Dt}} \frac{\partial \mathbf{r}}{\partial q_i}\right] dq_i$$

$$= \; \rho \, \left[\frac{\mathbf{D}}{\mathbf{Dt}}\left(\dot{r} \bullet \frac{\partial \mathbf{r}}{\partial q_i}\right) - \dot{r} \bullet \frac{\partial \dot{r}}{\partial q_i}\right] dq_i$$

Since $\frac{\partial \mathbf{r}}{\partial q_i} = \frac{\partial \dot{r}}{\partial \dot{q}_i}$ which is proved in equation (4-12) in Ying's advanced dynamics [1] and $\dot{r} = V$,

$$\text{L.H.S.} \; = \; \rho \left[\frac{\mathbf{D}}{\mathbf{Dt}}\left(\frac{\partial}{\partial \dot{q}_i} \frac{V^2}{2}\right) - \dot{r} \bullet \frac{\partial \dot{r}}{\partial q_i}\right] dq_i$$

$$= \; \rho \left[\frac{\mathbf{D}}{\mathbf{Dt}}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}\right] dq_i \quad (4)$$

where $T = \frac{V^2}{2}$ kinetic energy per unit mass. For the other term

$$\mathbf{F}.\frac{\partial \boldsymbol{r}}{\partial q_i} \, dq_i = \; Q_i \, dq_i \quad (5)$$

Where Q_i is called generalized force = $\mathbf{F}.\frac{\partial \boldsymbol{r}}{\partial q_i}$. Hence by combining equations (4) and (5) with (3), we have

$$\sum_i \left\{\rho \left[\frac{\mathbf{D}}{\mathbf{Dt}}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}\right] - Q_i\right\} dq_i = 0$$

Because all the q_i are independent, the coefficients must be zero. Therefore,

$$\rho \left[\frac{\mathbf{D}}{\mathbf{Dt}}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}\right] = \; Q_i \quad (6)$$

This is the Lagrangian equation for incompressible fluid in fluid mechanics.

Application

To illustrate the use of the equation three different coordinates are chosen, Cartesian, cylindrical and spherical. Since they are familiar to everybody, they can be recognized easily.

Cartesian coordinates

$$d\mathbf{r} = \mathbf{i} \, dx + \mathbf{j} \, dy + \mathbf{k} \, dz$$

$$\dot{\mathbf{r}} = \mathbf{I} \, \dot{x} + \mathbf{j} \, \dot{y} + \mathbf{k} \, \dot{z}$$

$$T = \frac{1}{2} \, V^2 = \frac{1}{2} \, (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Applying the Lagrangian equation

As $q_i = x$, we have

$$\rho \left[\frac{\mathbf{D}}{\mathbf{Dt}} \frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x}\right] = \; Q_x$$

$$\rho \, \frac{\mathbf{D}\dot{x}}{\mathbf{Dt}} = \; F_x$$

$$\rho\left(\frac{\partial u}{\partial t} + u \, \frac{\partial u}{\partial x} + v \, \frac{\partial u}{\partial y} + w \, \frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} \quad (7)$$

where $u = \dot{x}$, $v = \dot{y}$, $w = \dot{z}$ and P is pressure.

Similarly for $q_i = y$, we have

$$\rho \, \frac{\mathbf{D}\dot{y}}{\mathbf{Dt}} = \; F_y$$

$$\rho\left(\frac{\partial v}{\partial t} + u \, \frac{\partial v}{\partial x} + v \, \frac{\partial v}{\partial y} + w \, \frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} \quad (8)$$

For $q_i = z$, we have

$$\rho \, \frac{\mathbf{D}\dot{z}}{\mathbf{Dt}} = \; F_z$$

$$\rho\left(\frac{\partial w}{\partial t} + u \, \frac{\partial w}{\partial x} + v \, \frac{\partial w}{\partial y} + w \, \frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} \quad (9)$$

Cylindrical coordinates

$$d\mathbf{r} = \mathbf{e_R} \, d\mathbf{R} + \mathbf{e_\varphi} R \, d\varphi + \mathbf{k} \, d z$$

$$T = \frac{1}{2} \, V^2 = \frac{1}{2} \left[\dot{R}^2 + (R \, \dot{\varphi})^2 + \dot{z}^2\right]$$

For $q_i = R$, we have

$$\rho \left[\frac{\mathbf{D}}{\mathbf{Dt}} \frac{\partial T}{\partial \dot{R}} - \frac{\partial T}{\partial R}\right] = \; Q_R$$

$$\rho \left[\frac{\mathbf{D}}{\mathbf{Dt}} \dot{R} - R \, \dot{\varphi}^2\right] = \; Q_R$$

$$\rho \left[\frac{DV_R}{\mathbf{Dt}} - \frac{V_\varphi^2}{R}\right] = \; F_R = -\frac{\partial P}{\partial R} \quad (10)$$
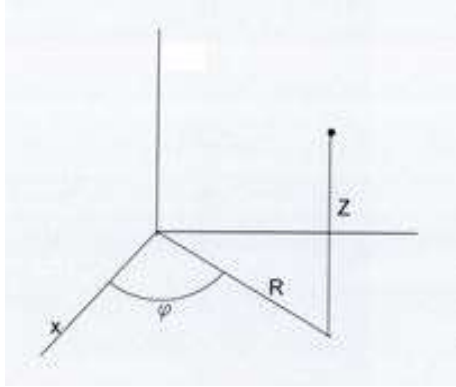
For $q_i = \varphi$, we have

Figure 1: Cylindrical coordinates.

$$\rho \left[\frac{D}{Dt} (R^2 \dot\varphi) \right] = Q_\varphi = F_\varphi R$$

Since $\frac{\partial r}{\partial \varphi} = e_\varphi R$

$$\rho \left[\frac{D}{Dt} \left(R\, V_\varphi \right) \right] = \rho R \left[\frac{DV_\varphi}{Dt} + \frac{V_R V_\varphi}{R} \right] = F_\varphi R \quad (11)$$

Similarly, for $q_i = z$, we have

$$\rho \left(\frac{D\dot z}{Dt} \right) = Q_z = F_z$$

$$\rho \frac{D V_z}{Dt} = -\frac{\partial P}{\partial z} \quad (12)$$

Where $V_R = \dot R$, $V_\varphi = R\dot\varphi$, $V_z = \dot z$ and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + V_R \frac{\partial}{\partial R} + \frac{V_\varphi}{R} \frac{\partial}{\partial \varphi} + V_z \frac{\partial}{\partial z}$$

Spherical coordinates

$$d\mathbf{R} = dr\, \mathbf{e}_r + r\, d\theta\, \mathbf{e}_\theta + r \sin\theta\, d\varphi\, \mathbf{e}_\varphi$$

$$\mathbf{V} = \dot r\, \mathbf{e}_r + r\, \dot\theta\, \mathbf{e}_\theta + r\dot\varphi \sin\theta\, \mathbf{e}_\varphi$$

$$T = \frac{1}{2} V^2 = \frac{1}{2} \left[\dot r^2 + \left(r\, \dot\theta \right)^2 + (r\, \dot\varphi \sin\theta)^2 \right]$$

$$\frac{\partial \mathbf{R}}{\partial r} = \mathbf{e}_r \qquad \frac{\partial \mathbf{R}}{\partial \theta} = r\, \mathbf{e}_\theta \qquad \frac{\partial \mathbf{R}}{\partial \varphi} = r \sin\theta\, \mathbf{e}_\varphi$$

For $q_i = r$, we have

$$\frac{\partial T}{\partial \dot r} = \dot r \qquad \frac{\partial T}{\partial r} = r\, \dot\theta^2 + r\, \dot\varphi^2 (\sin\theta)^2$$
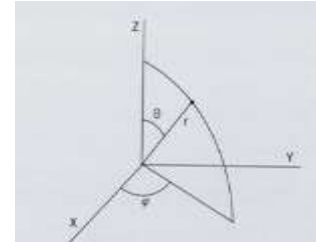

Figure 2: Spherical coordinates.

$$\rho \left\{ \frac{D\dot r}{Dt} - \left[r\dot\theta^2 + r\, (\dot\varphi \sin\theta)^2 \right] \right\} = Q_r$$

$$\rho \left[\frac{D\, V_r}{Dt} - \frac{1}{r} \left(V_\theta^2 + V_\varphi^2 \right) \right]$$

$$= Q_r = -\frac{\partial P}{\partial r} + \rho \mathbf{g} . \mathbf{e}_r \quad (13)$$

For $q_i = \theta$, we have

$$\frac{\partial T}{\partial \dot\theta} = r^2 \dot\theta \qquad \frac{\partial T}{\partial \theta} = r^2 \dot\varphi^2 \sin\theta \cos\theta$$

$$\rho \left[\frac{D}{Dt} \left(r^2 \dot\theta \right) - r^2 \dot\varphi^2 \sin\theta \cos\theta \right] = Q_\theta$$

$$\rho \left[\frac{D}{Dt} (r\, V_\theta) - V_\varphi^2 \cot\theta \right] = F_\theta\, r$$

$$\rho\, r \frac{DV_\theta}{Dt} + \rho\, V_\theta V_r - \rho\, V_\varphi^2 \cot\theta = F_\theta\, r$$

$$\rho \frac{DV_\theta}{Dt} + \rho \frac{V_\theta V_r}{r} - \rho \frac{V_\varphi^2}{r} \cot\theta = -\frac{1}{r} \frac{\partial P}{\partial \theta} + \rho\, \mathbf{g} \bullet \mathbf{e}_\theta \quad (14)$$

Similarly for $q_i = \varphi$, we have

$$\rho \frac{D}{Dt} [(r \sin\theta)^2 \dot\varphi] = Q_\varphi = \mathbf{F} \bullet \frac{\partial \mathbf{R}}{\partial \varphi}$$

$$= \mathbf{F} \bullet r \sin\theta\, \mathbf{e}_\varphi = F_\varphi r \sin\theta$$

$$\rho \frac{D}{Dt} \left(r \sin\theta\, V_\varphi \right)$$

$$= \rho\, r \sin\theta \frac{DV_\varphi}{Dt} + \rho\, V_r \sin\theta\, V_\varphi + \rho\, r \cos\theta\, \dot\theta\, V_\varphi$$

$$= F_\varphi\, r \sin\theta$$

$$\rho \frac{DV_\varphi}{Dt} + \rho \frac{V_r V_\varphi}{r} + \rho \frac{V_\varphi V_\theta}{r} \cot\theta$$

$$= F_\varphi = -\frac{1}{r \sin\theta} \frac{\partial P}{\partial \varphi} + \rho\, \mathbf{g} \bullet \mathbf{e}_\varphi \quad (15)$$

Where $V_r = \dot r$, $V_\theta = r\, \dot\theta$, $V_\varphi = r \sin\theta\, \dot\varphi$,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + V_r \frac{\partial}{\partial r} + \frac{V_\theta}{r} \frac{\partial}{\partial \theta} + \frac{V_\varphi}{r \sin\theta} \frac{\partial}{\partial \varphi}$$

CONCLUSION

This paper may be considered as the first step for the use of Lagrangian equation in fluid mechanics. An inviscid incompressible fluid is considered here, more general form of the Lagrangian equation can be developed for more general kind of fluid. Now the door is open, further development and application of the equation can be very fruitful to scientists in fluid mechanics.

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